Linear systems – Resit exam

Resit exam 2022–2023, Monday 10 July 2023, 15:00 – 17:00

Instructions

1. The use of books, lecture notes, or (your own) notes is not allowed.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1

(6 + 10 + 6 = 22 points)

Consider the model of population dynamics given by

$$\begin{aligned}\dot{x}_1(t) &= (\beta_1 - F(x(t)))x_1(t),\\ \dot{x}_2(t) &= (\beta_2 - F(x(t)))x_2(t),\\ \dot{x}_3(t) &= (\beta_3 - F(x(t)))x_3(t),\end{aligned}$$

where $x_i(t) \in \mathbb{R}$, $i = 1, 2, 3$ denote the populations of three species and $x = [\, x_1 \; x_2 \; x_3 \,]^{\mathrm{T}}$. Here,

$$F(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3,$$

for real parameters $\alpha_i > 0$, $i = 1, 2, 3$, denotes the total burden on the environment and $\beta_i > 0$, $i = 1, 2, 3$, denote the natural growth rates for each species. They are assumed to satisfy

$$\beta_1 > \beta_2 > \beta_3 > 0.$$

(a) Apart from the trivial equilibrium $\bar{x} = 0$, show that any other equilibrium point is necessarily of the form $\bar{x}_i > 0$ for some $i \in \{1, 2, 3\}$ and $\bar{x}_j = 0$ for $j \neq i$.

(b) Consider the equilibrium

$$\bar{x} = \begin{bmatrix} \frac{\beta_1}{\alpha_1} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

and linearize the system around this equilibrium point.

(c) Is the linearized system (asymptotically) stable?

Problem 2

(18 points)

Consider a linear system characterized through the transfer function

$$T(s) = \frac{s+2}{s^4 + as^3 + 4s^2 + 2as + 2a},$$

where $a \in \mathbb{R}$. Give the values of a for which the system is externally stable.

Problem 3

(4 + 4 + 10 + 10 = 28 points)

Consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad \text{with} \quad A = \begin{bmatrix} 5 & -1 \\ 6 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 \end{bmatrix},$$

where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$.

(a) Compute the transfer function of the system.

(b) Verify that the system is observable.

(c) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(d) Use the matrix T from (c) to design a stable state observer

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + G(y(t) - C\xi(t))$$

such that the resulting error dynamics satisfies $\sigma(A - GC) = \{-2, -3\}$.

Problem 4

(14 + 8 = 22 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$.

(a) Let $m = 1$. Show that the system cannot be controllable if there are two linearly independent eigenvectors of A^{T} corresponding to the same eigenvalue $\lambda \in \sigma(A)$.

(b) Generalize the above by allowing m to be any positive integer. In particular, show that the system cannot be controllable if there are $m + 1$ linearly independent eigenvectors of A^{T} corresponding to the same eigenvalue $\lambda \in \sigma(A)$.

(10 points free)